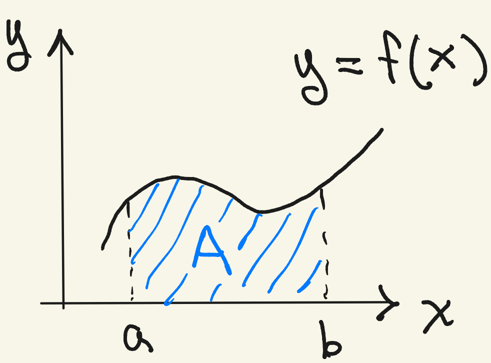
§15.1 Double Integrals -

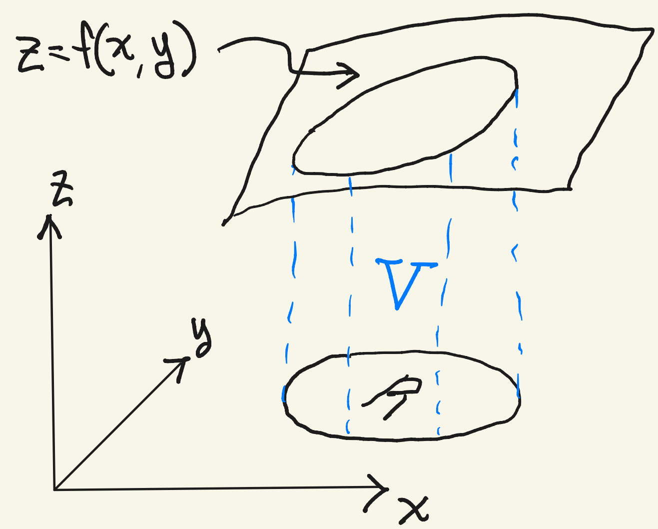
- **Defn in Words**: $\iint_R f(x,y)\,dA \equiv$ "The volume of the region under the graph of f above the (x,y)-plane."

- **Generalizes**: $\int_a^b f(x)\,dx \equiv$ "The area of the region under the graph of f above the interval [a,b]."

- **Picture**



$$A = \int_a^b f(x)\,dx$$

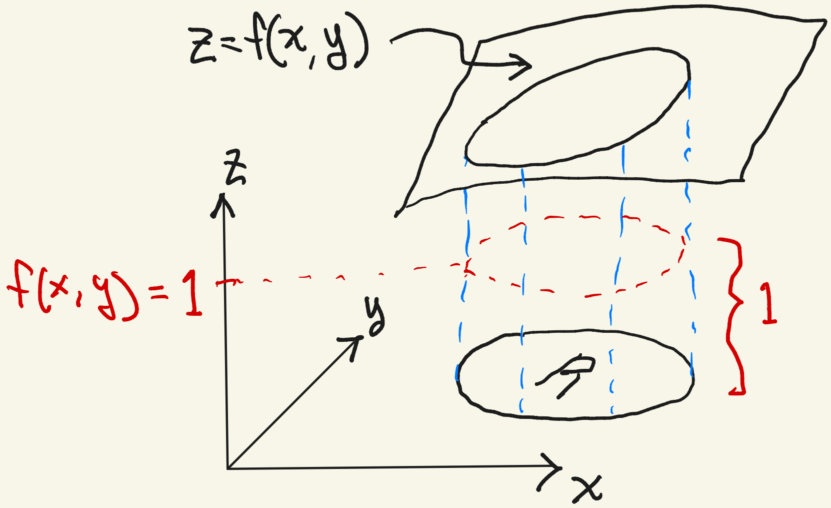


$$V = \iint_R f(x,y)\,dA$$

- **Note**: If $f(x,y) = 1$, then

$$\iint_R 1 \cdot dA = \text{Area of } R$$

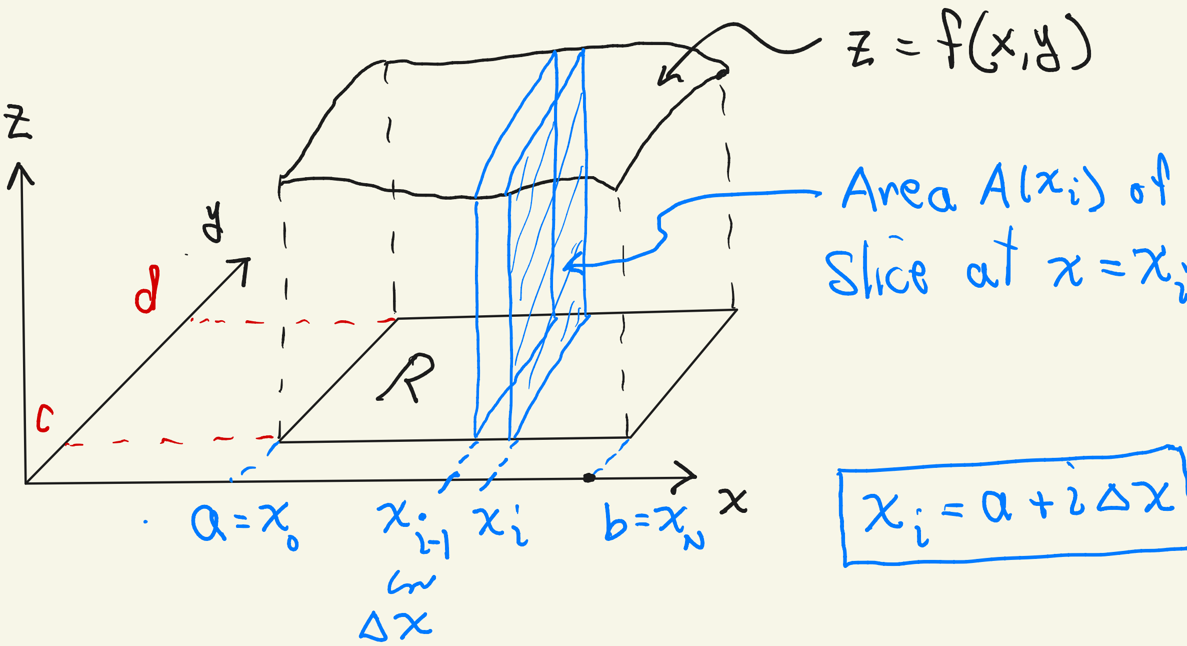
That is... if $f(x,y)=1$



$$\iint_R 1\cdot dA = \text{Area of } R$$

- **Simplest Case:** R is a Rectangle $R=[a,b]\times[c,d]$

Notation: $R=[a,b]\times[c,d] \equiv \{(x,y) : x\in[a,b] \ \& \ y\in[c,d]\}$



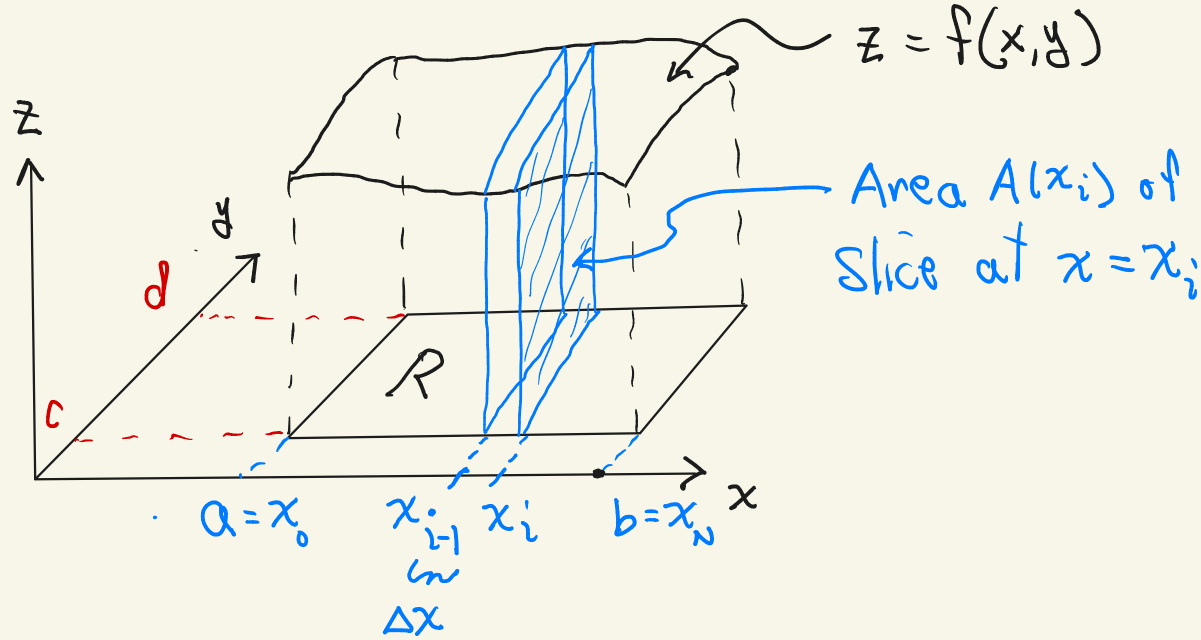
$$\boxed{x_i = a + i\Delta x}$$

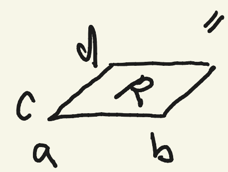
- **Define** a **mesh:** $a=x_0<x_1<\cdots<x_i<\cdots<x_N=b$

Always the **same way:** $N = \#$ of mesh points

$\Delta x = $ dist betw points $= x_i - x_{i-1} = \dfrac{b-a}{N}$, $x_i = a + i\Delta x$

- Now... construct the volume from an approximate volume based on slices of Area



$\iint_R f(x,y)\,dA$ = "Vol under graph of f above "

Let $A(x)$ = "Area of slice at x" $= \int_c^d f(x,y)\,dy$

$$\text{Vol} \approx \underbrace{\sum_{i=1}^{N} A(x_i)\Delta x}_{\text{Riemann Sum}} = \int_a^b A(x)\,dx = \int_a^b \underbrace{\int_c^d f(x,y)\,dy}_{A(x)}\,dx$$

Conclude: $$\iint_R f(x,y)\,dA = \int_a^b \underbrace{\int_c^d f(x,y)\,dy}_{\text{Integrate wrt } y \text{ at fixed } x \text{ to get a function of } x \text{ alone}}\,dx$$

Theorem: (Fubini – I) when R is a rectangle

$R = [a,b] \times [c,d]$

$a \le x \le b$

$c \le y \le d$

we have:

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Proof: Put the mesh on y-axis

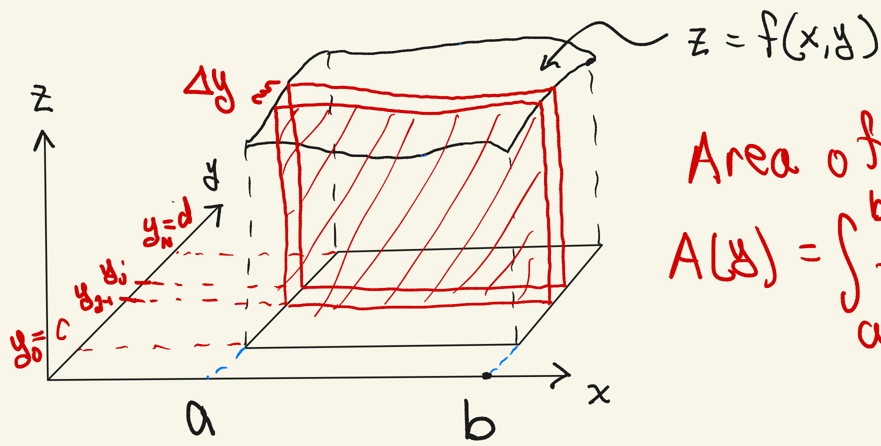
$$c = y_0 < y_1 < \cdots < y_j < \cdots < y_N = d$$

$$\Delta y = \frac{d-c}{N}, \quad y_j = c + j\Delta y$$

So
$$\iint_R f(x,y)\,dA \approx \sum_{j=1}^{N} A(y_j)\Delta y \xrightarrow[\Delta y \to 0]{} \int_c^d A(y)\,dA$$

$$= \int_c^d \underbrace{\int_a^b f(x,y)\,dx}_{A(y)}\,dy$$

Picture



Area of slice

$$A(y) = \int_a^b f(x,y)\,dx$$

Example: $f(x,y) = xy$, $0 \le x \le 1$, $1 \le y \le 2$

$R = [0,1] \times [1,2]$ evaluate $I = \iint_R f(x,y)\, dA$ two ways –

Solution:

(1) $$I = \int_0^1 \underbrace{\int_1^2 xy\, dy}_{x\frac{y^2}{2}\Big]_{y=1}^{y=2}}\, dx = \int_0^1 x \frac{y^2}{2}\Big]_{y=1}^{y=2} dx = \int_0^1 x \underbrace{\left(\frac{4}{2} - \frac{1}{2}\right)}_{3/2} dx$$

$$= \frac{3}{2}\int_0^1 x\, dx = \frac{3}{2}\frac{x^2}{2}\Big]_0^1 = \frac{3}{2}\frac{1}{2} = \boxed{\frac{3}{4}}$$

(2) $$I = \int_1^2 \underbrace{\int_0^1 xy\, dx}_{\frac{x^2}{2}y\Big]_{x=0}^{x=1}}\, dy = \int_1^2 \frac{x^2}{2} y \Big]_{x=0}^{x=1} dy = \int_1^2 \frac{1}{2} y\, dy$$

$$= \frac{1}{2}\frac{y^2}{2}\Big]_1^2 = \frac{1}{2}\left(\frac{2^2}{2} - \frac{1^2}{2}\right) = \boxed{\frac{3}{4}}$$

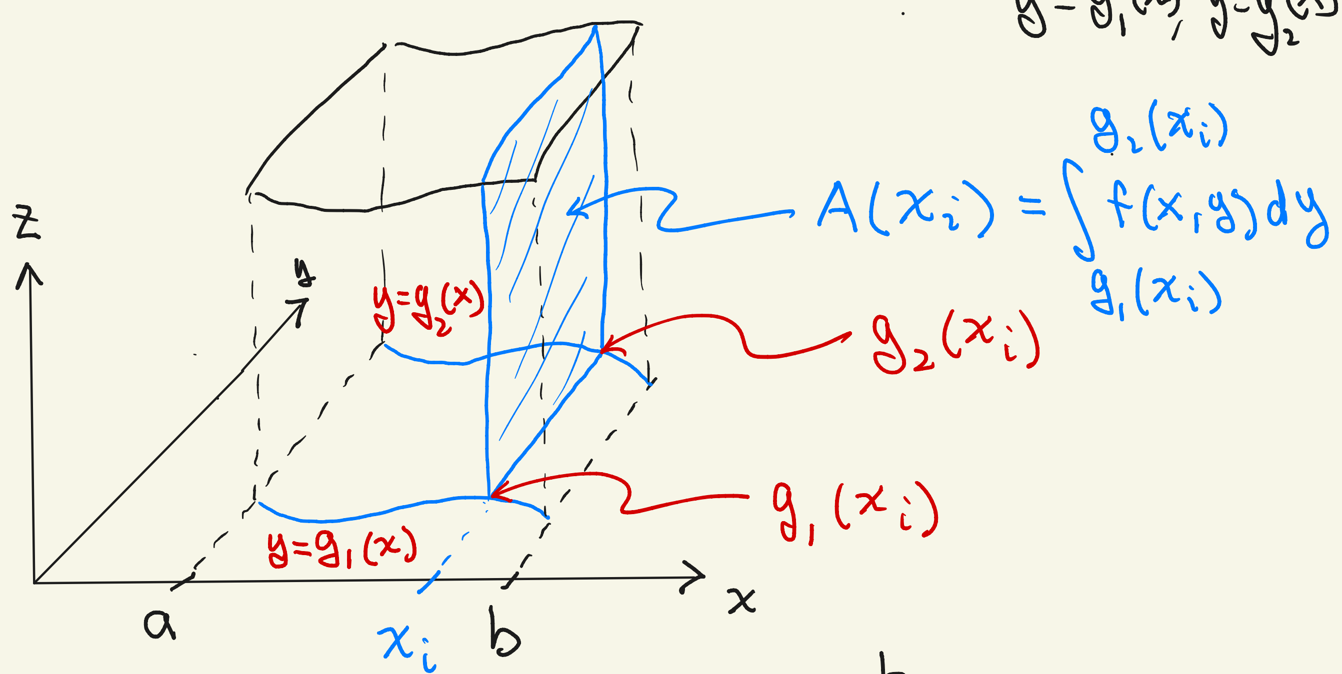
Note: Both iterated integrals are solved by different arithmatic but yield same ans!

Note: In general, one iterate may be solvable by Math 21B methods, when the other might not!

- When R is not a rectangle, the procedure for changing the order of integration is more complicated!

General Case of General Slicing

R = region in (x,y)-plane bounded by $x=a$, $x=b$, $y=g_1(x)$, $y=g_2(x)$



$$\iint_R f(x,y)\,dA \approx \underbrace{\sum_{i=1}^{N} A(x_i)\Delta x}_{\text{Riemann Sum}} \xrightarrow[N\to\infty]{} \int_a^b \underbrace{A(x)}_{\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy}\,dx$$

$$= \boxed{\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx}$$

This is a formula for computing the volume

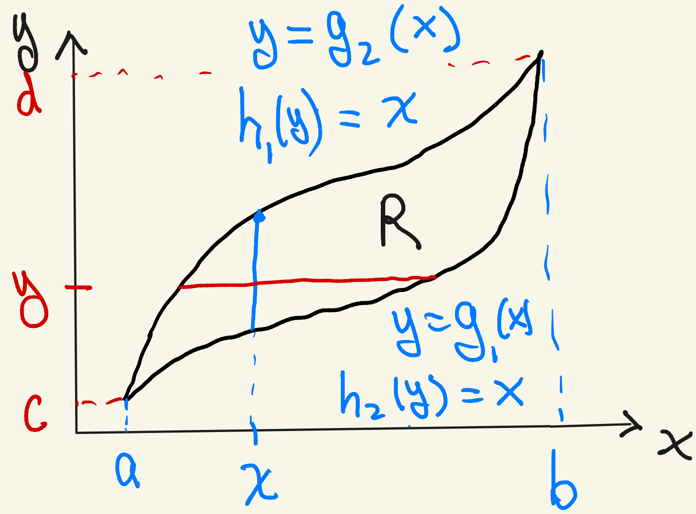
Note: You might not be able to slice the other way!

☒ General Case of Fubini's Thm when you can iterate an integral <u>both</u> <u>ways</u>

To iterate, you only need R in (x,y)-plane

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$$

$$A(y) = \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx$$



Then: $I = \iint_R f(x,y)\,dA$ can be iterated 2-ways:
(sliced)

$$I = \int_a^b A(x)\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

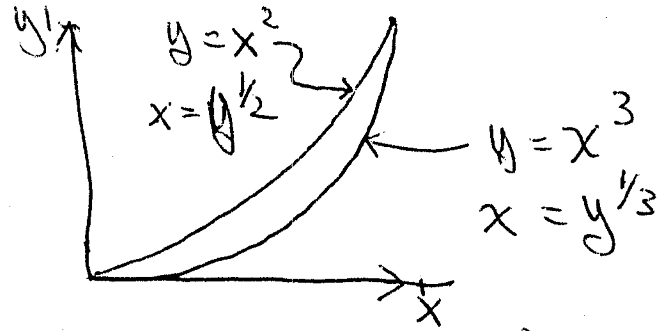
Recall: Riemann Integral requires f be <u>continuous</u>

$$I = \int_c^d A(y)\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Note: $\boxed{\begin{array}{l} y = g_2(x) \\ h_1(y) = x \end{array}} \Leftrightarrow g_2^{-1} = h_1$ $\quad \boxed{\begin{array}{l} y = g_1(x) \\ h_2(y) = x \end{array}} \Leftrightarrow g_1^{-1} = h_2$

Ex: Evaluate $\iint_R x \, dA$ where R is region betw ~~graphs of~~ curves $y = x^3$ & $y = x^2$ $\boxed{\text{2-ways}}$

Picture:



(1) $\int_0^1 \int_{x^3}^{x^2} x \, dy \, dx = \int_0^1 x \, y \Big]_{y=x^3}^{y=x^2} dx = \int_0^1 x^3 - x^4 \, dx = \frac{x^4}{4} - \frac{x^5}{5}\Big]_0^1$

$= \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$

(2) $\int_0^1 \int_{\sqrt{y}}^{\sqrt[3]{y}} x \, dx \, dy = \int_0^1 \frac{x^2}{2}\Big]_{x=\sqrt{y}}^{x=\sqrt[3]{y}} dy = \int_0^1 \frac{(y^{1/3})^2}{2} - \frac{(y^{1/2})^2}{2}$

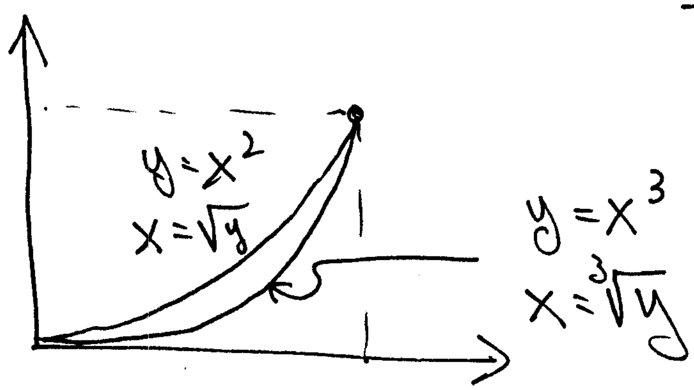
$= \int_0^1 \frac{y^{2/3}}{2} - \frac{y}{2} \, dy = \frac{3}{5}\frac{y^{5/3}}{2} - \frac{y^2}{4}\Big]_0^1 = \frac{3}{10} - \frac{1}{4}$

$= \frac{12-10}{40} = \frac{1}{20}$

Use dbl Int

Ex: Find the area between the curves $y = x^3$ and $y = x^2$ <u>3-ways</u>

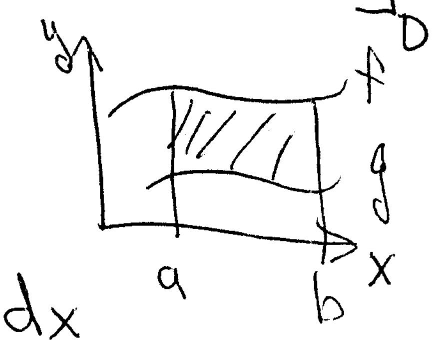
<u>Picture</u>:



① $$\int_0^1 \int_{x^3}^{x^2} 1 \cdot dy\,dx = \int_0^1 [x^2 - x^3]\,dx = \frac{x^3}{3} - \frac{x^4}{4}\Big]_0^1 = \frac{1}{12}$$

② $$\int_0^1 \int_{\sqrt{y}}^{\sqrt[3]{y}} 1 \cdot dx\,dy = \int_0^1 [y^{1/3} - y^{1/2}]\,dy = \frac{3y^{4/3}}{4} - \frac{2y^{3/2}}{3}\Big]_0^1 = \frac{1}{12}$$

③ Check from 21A: Area betw f & g



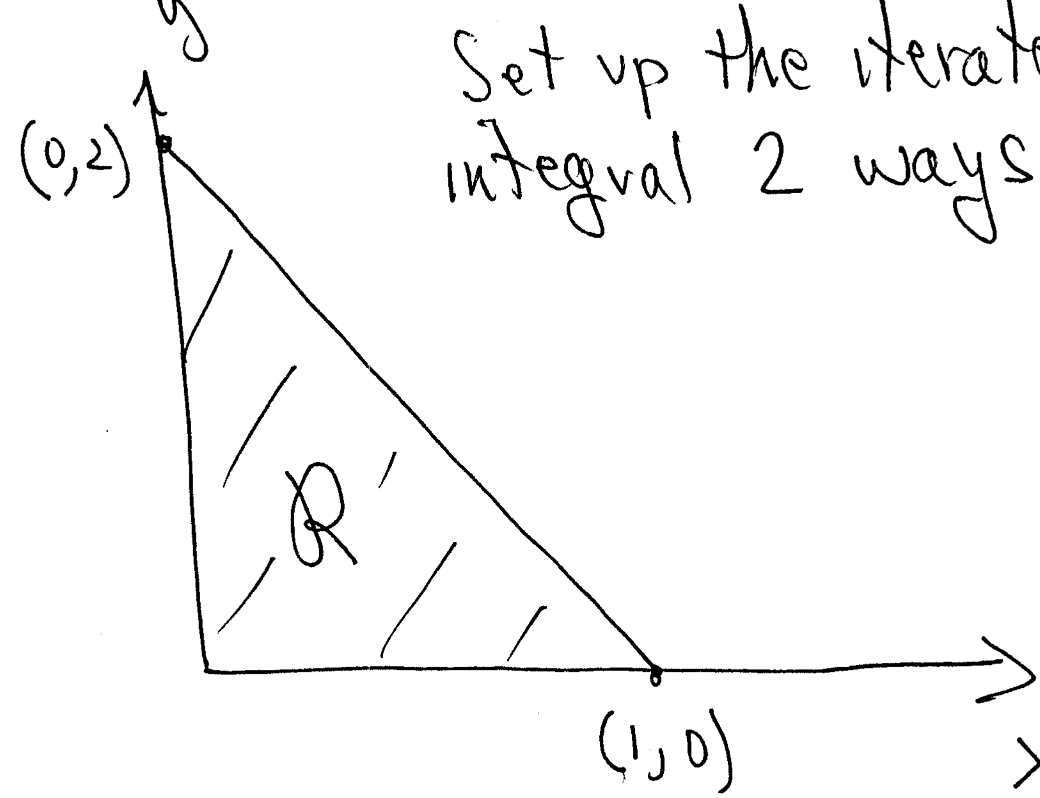
$$\text{Area} = \int_a^b [f(x) - g(x)]\,dx = \int_0^1 x^2 - x^3\,dx$$

$$= \frac{x^3}{3} - \frac{x^4}{4}\Big]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \checkmark$$

Ex Find the volume under the graph of $f(x,y) = xy$ above the region R bounded by the x-axis, y-axis & the line that passes thru $(1,0)$ & $(0,2)$

(2-ways)

Picture:



Set up the iterated integral 2 ways -

Soln

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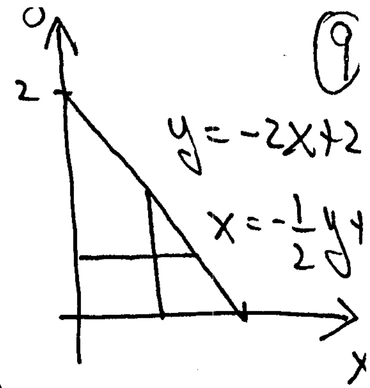


$$2x = -y + 2$$

- $y = -2x + 2$, $\quad x = -\frac{1}{2}y + 1$

- $$\int_0^1 \int_0^{-2x+2} xy \, dy \, dx = \int_0^2 \int_0^{-\frac{1}{2}y+1} xy \, dx \, dy$$

$$LHS = \int_0^1 x \left.\frac{y^2}{2}\right]_0^{-2x+2} dx = \int_0^1 x \left[\frac{(-2x+2)^2}{2} - 0\right] dx$$

$$= \int_0^1 x \left[\frac{\cancel{4}^{\,2}x^2 + (-\cancel{8}^{\,-4}x) + \cancel{4}^{\,2}}{\cancel{2}}\right] dx = \int_0^1 2x^3 - 4x^2 + 2x \, dy$$

$$= 2\frac{x^4}{4} - 4\frac{x^3}{3} + 2\frac{x^2}{2}\Big]_0^1 = \frac{2}{4} - \frac{4}{3} + \frac{2}{2} = \frac{3}{2} - \frac{4}{3} = \frac{1}{6}$$

$$RHS = \int_0^2 \int_0^{-1/2 y+1} xy \, d\cancel{x} \, d\cancel{y} = \int_0^2 y \int_0^{-\frac{1}{2}y+1} x \, dx \, dy$$

$$= \int_0^2 y \left.\frac{x^2}{2}\right]_0^{-\frac{1}{2}y+1} dy = \int_0^2 y \frac{\left(-\frac{1}{2}y+1\right)^2}{2} dy = \int_0^2 y \left[\frac{+\frac{1}{4}y^2 - y + 1}{2}\right] dy$$

$$= \int_0^2 +\frac{1}{8}y^3 - \frac{y^2}{2} + \frac{y}{2} \, dy = \frac{y^4}{8\cdot 4} - \frac{y^3}{3\cdot 2} + \frac{y^2}{2\cdot 2}\Big]_0^2$$

$$= \frac{2^4}{8\cdot 4} - \frac{2^3}{3\cdot 2} + \frac{2^2}{2\cdot 2} = \frac{1}{2} - \frac{4}{3} + 1 = \frac{3}{2} - \frac{4}{3} = \frac{1}{6}$$

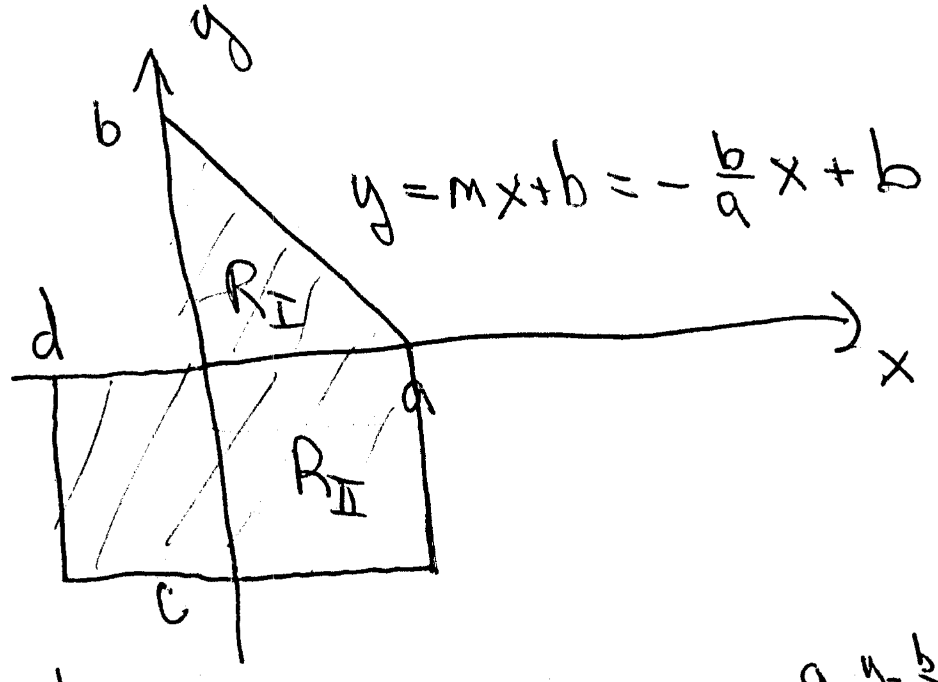
Ex: Some times the region R_{xy} has to be broken up to iterate the integral —

Ex ~~Find~~ set up/iterate the integral

$$\iint_{R_{xy}} f(x,y)\,dA \qquad \text{where } f(x,y)=\tan^{-1}(xy)$$

where:

$$\iint_{R_{xy}} f = \iint_{R_I} f + \iint_{R_{II}} f$$



$$A(y)=\int_0^{y=-\frac{b}{a}x+b} f(x,y)\,d\cancel{x} \quad \text{in } R_I \qquad \iint_{R_1} f\,dA=\int_0^a\int_0^{y=-\frac{b}{a}x+b} f(x,y)\,dx\,dy$$

$$A(y)=\int_d^a f(x,y)\,dx \quad \text{in } R_{II} \qquad \iint_{R_2} f\,dA=\int_c^0\int_d^a f(x,y)\,dx\,dy$$

Theorem: Integrals can in general be broken up —

① $$\iint_{R_I} f(x,y)\,dA + \iint_{R_{II}} f(x,y)\,dA = \iint_{R_I \cup R_{II}} f(x,y)\,dA$$

So long as R_I & R_{II} don't overlap

ie if $R_I \cap R_{II} = \emptyset$

② $$\iint_{R_{xy}} f(x,y) + g(x,y)\,dA = \iint_{R_{xy}} f(x,y)\,dA + \iint_{R_{xy}} g(x,y)\,dA$$

Eg: $$\iint_{R_{xy}} x^2 + \sin xy\,dA = \iint_{R_{xy}} x^2\,dA + \iint_{R_{xy}} \sin xy\,dA$$

③ $$\iint_{R_{xy}} k\,f(x,y)\,dA = k \iint_{R_{xy}} f(x,y)\,dA$$ "you can pull consts thru $\int$-sign"

If $f(x,y) \geq g(x,y)$ $\forall (x,y) \in R_{xy}$, then (11)

(4) $$\iint_{R_{xy}} f(x,y)\, dA \geq \iint_{R_{xy}} g(x,y)\, dA$$

cont Theorem: Fubini I: R $a \le x \le b$, $c \le y \le d$

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Ex: We had: $\int_1^2 \int_1^3 xy\,dy\,dx = 6$

Check: $\int_1^3 \int_1^2 xy\,dx\,dy = \int_1^3 y\left[\int_1^2 x\,dx\right] dy$

$$= \int_1^3 y \frac{x^2}{2}\Big]_1^2 dy = \int_1^3 \left(\frac{4}{2} - \frac{1}{2}\right) y\,dy$$

$$= \frac{3}{2} \frac{y^2}{2}\Big]_1^3 = \frac{3}{2}\left(\frac{9}{2} - \frac{1}{2}\right) = \frac{3}{\cancel{2}} \cdot \frac{\cancel{8}}{2} = \boxed{6}$$

✓

cont

ii Radius of gyration – (of a body about an axis)

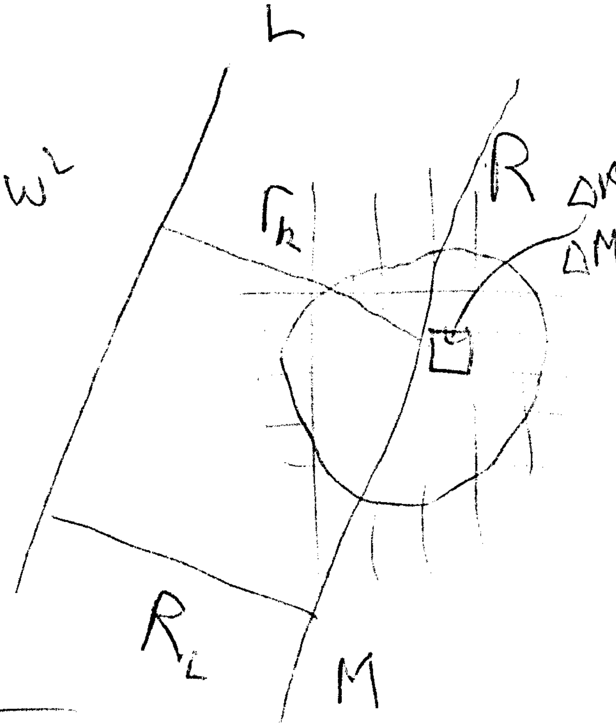
$$R_L = \sqrt{I_L/M}$$

Explain. R_L is radius at which you could put all the mass st you get same KE of rotation.

I.e. want $KE = \frac{1}{2} I_L \omega^2 = \frac{1}{2} M R^2 \omega^2$

$$I_L = MR^2$$

$$\frac{I_L}{M} = R$$



$KE = \frac{1}{2} M v^2 = \frac{1}{2} M R^2 \omega^2$

Find radius of gyration for above –

$$R_y = \sqrt{\frac{I_y}{M}}$$

$\underline{\underline{Ex}}$ Draw the region of integration & reverse order: $\displaystyle\int_0^{\ln 2}\int_{e^y}^{2} dx\,dy$

$$\int_0^{\ln 2} 2-e^y\,dy = 2\ln 2 - e^y\Big]_0^{\ln 2} = 2\ln 2 - e^{\ln 2} + e^0 = 2\ln 2 - 2 + 1 = 2\ln 2 - 1$$

$\underline{\underline{\text{Soln}}}$: $\displaystyle "\int_{e^y}^{2} dx" = \int_{x=e^y}^{x=2} dx$

"integrate 1st wrt x from $x=e^y$ to $x=2$"

$x=e^y \Leftrightarrow \ln x = y$

$$= \int_1^2\int_0^{\ln x} dy\,dx = \int_1^2 \ln x\,dx$$

$u=\ln x$

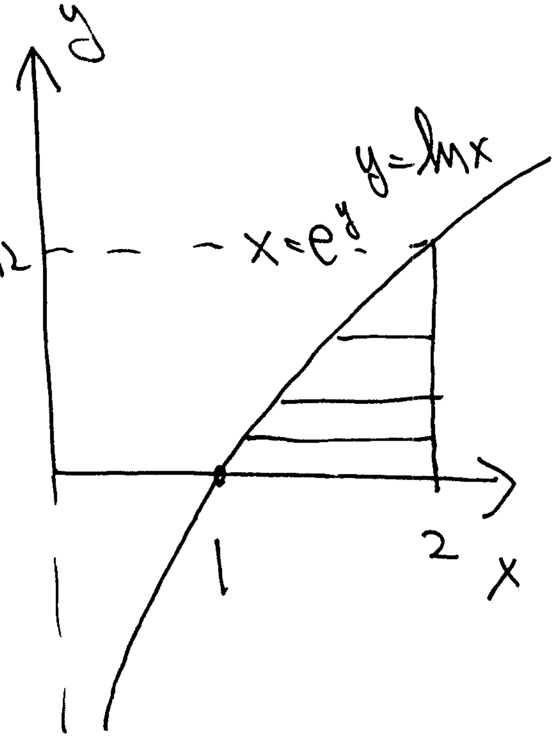
$dv=dx$

$du=\frac{1}{x}dx$

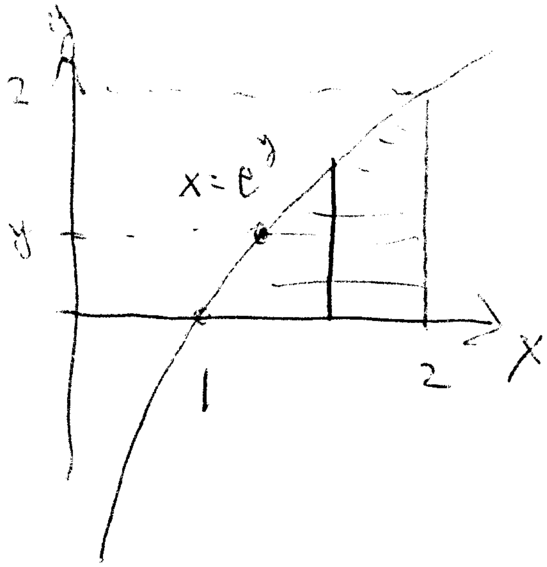
$v=x$

$$= x\ln x\Big]_1^2 - \int_1^2 1\,dx$$

$$= 2\ln 2 - 2 - 1 = 2\ln 2 - 1$$



$$\int_0^{\ln 2}\int_{x=e^y}^{x=2} dx\,dy$$



$$x = e^y \iff \ln x = y$$

$$= \int_1^2 \int_{y=0}^{y=\ln x} dy\,dx = \int_1^2 y\Big]_{y=0}^{y=\ln x} dx = \int_1^2 (\ln x)\,dx$$

$u = \ln x \quad dv = dx$

$du = \frac{dx}{x} \quad v = x$

$= uv\Big] - \int v\,du$

$$= x\ln x\Big]_1^2 - \int_1^2 dx$$

$$= 2\ln 2 - 1\underbrace{\cancel{\ln 1}}_{0} - 1 = 2\ln 2 - 1$$

Other way: $\displaystyle\int_0^{\ln 2}\int_{e^y}^{2} dx\,dy = \int_0^{\ln 2} x\Big]_{e^y}^{2} dy = \int_0^{\ln 2} 2 - e^y\,dy$

$$= 2\ln 2 - \int_0^{\ln 2} e^y\,dy = 2\ln 2 - e^y\Big]_0^{\ln 2} = 2\ln 2 - 2 + 1$$

$$= 2\ln 2 - 1 \checkmark$$